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Broadcasting a Common Message with Variable-Length Stop-Feedback Codes

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Abstract—We investigate the maximum coding rate achievable over a two-user broadcast channel for the scenario where a common message is transmitted using variable-length stop-feedback codes. Specifically, upon decoding the common message, each decoder sends a stop signal to the encoder, which transmits continuously until it receives both stop signals. For the point-to-point case, Polyanskiy, Poor, and Verdú (2011) recently demonstrated that variable-length coding combined with stop feedback significantly increases the speed at which the maximum coding rate converges to capacity. This speed-up manifests itself in the absence of a square-root penalty in the asymptotic expansion of the maximum coding rate for large blocklengths, a result a.k.a. *zero dispersion*. In this paper, we show that this speed-up does not necessarily occur for the broadcast channel with common message. Specifically, there exist scenarios for which variable-length stop-feedback codes yield a positive dispersion.

I. INTRODUCTION

We consider the setup where an encoder wishes to convey a common message over a broadcast channel with noiseless feedback to two decoders. Similarly to the single-decoder (SD) case, noiseless feedback combined with fixed-blocklength codes does not improve capacity, which is given by [1, p. 126]

$$C = \sup_P \min\{I(P, W_1), I(P, W_2)\}. \quad (1)$$

Here, W_1 and W_2 denote the channels to decoder 1 and 2, respectively, and the supremum is over all input distributions P . For the case when there is no feedback, the speed at which C is approached as the blocklength n increases is of the order $1/\sqrt{n}$ [2] (same as in the SD case). The constant factor associated to the $1/\sqrt{n}$ term is commonly referred to as channel *dispersion*.

For the SD case, noiseless feedback combined with variable-length codes improve significantly the speed of convergence to capacity. Specifically, it was shown in [3] that

$$\frac{1}{l} \log \widetilde{M}_f^*(l, \epsilon) = \frac{\widetilde{C}}{1 - \epsilon} - \mathcal{O}\left(\frac{\log l}{l}\right) \quad (2)$$

where l stands for the average blocklength (average transmission time), $\widetilde{M}_f^*(l, \epsilon)$ is the maximum number of codewords in the SD case, and \widetilde{C} denotes the corresponding capacity. One sees from (2) that no square-root penalty occurs (zero dispersion), which implies a fast convergence to the asymptotic limit. This fast convergence is demonstrated numerically in [3] by means of nonasymptotic bounds. Variable-length stop-feedback (VLSF) codes, i.e., coding schemes where the feedback is used only to stop transmissions, are sufficient to achieve (2).

The purpose of this paper is to investigate whether a similar result holds for the broadcast channel with common message.

Contribution: We consider the subclass of discrete memoryless broadcast channels for which $I(P, W_1)$ and $I(P, W_2)$ are maximized by the same input distribution P^* , which we assume to be unique. In this case, $C = \min\{I(P^*, W_1), I(P^*, W_2)\}$. Focusing on the case when VLSF codes are used, we obtain nonasymptotic achievability and converse bounds on the maximum number of codewords $M_{sf}^*(l, \epsilon)$ with average blocklength l that can be transmitted with reliability $1 - \epsilon$. Here, the subscript “sf” stands for stop feedback. By analyzing these bounds in the large- l regime, we prove that when the two subchannels are independent and have the same capacity and the same dispersion, and when $\epsilon \leq 0.1968$, the asymptotic expansion of $M_{sf}^*(l, \epsilon)$ contains a square-root penalty (see (18) and (22) for a precise statement of this result). Hence, the fast convergence to the asymptotic limit experienced in the SD case cannot be expected.

The intuition behind this result is as follows: in the SD case, the stochastic variations of the information density that result in the square-root penalty can be virtually eliminated by using variable-length coding with stop-feedback. Indeed, decoding is stopped after the information density exceeds a certain threshold, which yields only negligible stochastic variations. In the broadcast setup, however, the stochastic variations in the difference between the stopping times at the two decoders make the square-root penalty reappear. Note that our result does not necessarily imply that feedback is useless. It only shows that VLSF codes cannot be used to speed-up convergence to the same level as in the SD case.

Proof techniques: The achievability bound is an extension of [3, Th 3]; the converse bound is based on an optimal stopping problem, where the probability that the stopping time exceeds a given threshold is minimized under a constraint on the “stopped” information density process. The asymptotic analysis of the converse bound relies on Hoeffding’s inequality and on the Berry-Esseen central limit theorem, whereas the asymptotic analysis of the achievability bound relies on asymptotic results for random walks [4] and on a Berry-Esseen-type theorem that holds for random summations [5].

Notation: Upper case, lower case, and calligraphic letters denote random variables (RV), deterministic quantities, and sets, respectively. The probability density function of a standard Gaussian RV is denoted by $\phi(x)$. Furthermore, $\Phi(x) \triangleq 1 - Q(x)$ is its cumulative distribution, where $Q(x)$ is the Q-function. We let x^+ and x^- denote $\max(0, x)$ and $\min\{0, x\}$, respectively. Throughout the paper, the index k belongs always

to the set $\{1, 2\}$, although this is sometimes omitted. Furthermore, $\bar{k} \triangleq 3 - k$. We adopt the convention that $\sum_{i=j}^{j-1} a_i = 0$ for all $\{a_i\}$ and all integers j . We use “c” to denote a finite nonnegative constant. Its value may change at each occurrence. Finally, \mathbb{N} denotes the set of positive integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

II. SYSTEM MODEL

A common-message discrete memoryless broadcast channel with two decoders is defined by the finite input alphabet \mathcal{X} and the finite output alphabets \mathcal{Y}_k , along with the stochastic matrices W_k , where $W_k(y_k|x)$ denotes the probability that $y_k \in \mathcal{Y}_k$ is observed at decoder k given $x \in \mathcal{X}$. We assume that the outputs at each time i are conditionally independent given the input, i.e.,

$$P_{Y_{1,i}, Y_{2,i}|X_i}(y_{1,i}, y_{2,i}|x_i) \triangleq W_1(y_{1,i}|x_i)W_2(y_{2,i}|x_i). \quad (3)$$

Define the set of probability distributions on \mathcal{X} by $\mathcal{P}(\mathcal{X})$. Let $P \times W_k : (x, y_k) \rightarrow P(x)W_k(y_k|x)$ denote the joint distribution of input and output at decoder k , and let $PW_k : y_k \rightarrow \sum_{x \in \mathcal{X}} P(x)W_k(y_k|x)$ denote the marginal distribution on \mathcal{Y}_k . For every $P \in \mathcal{P}(\mathcal{X})$, the information density is defined as

$$i_{P, W_k}(x^n; y_k^n) \triangleq \sum_{i=1}^n \log \frac{W_k(y_{k,i}|x_i)}{PW_k(y_{k,i})}. \quad (4)$$

We let $I(P, W_k) \triangleq \mathbb{E}_{P \times W_k}[i_{P, W_k}(X; Y_k)]$ be the mutual information, $V(P, W_k) \triangleq \text{Var}_{P \times W_k}[i_{P, W_k}(X; Y_k)]$ be the (unconditional) information variance, and $T(P, W_k) \triangleq \mathbb{E}_{P \times W_k}[|i_{P, W_k}(X; Y_k) - I(P, W_k)|^3]$ be the third absolute moment of the information density. We restrict ourselves to the case, where there exists a unique probability distribution $P^* \in \mathcal{P}(\mathcal{X})$ that maximizes simultaneously both $I(P, W_1)$ and $I(P, W_2)$. In this case, the capacity is given by

$$C \triangleq \min\{C_1, C_2\} \quad (5)$$

where $C_k \triangleq I(P^*, W_k)$. The corresponding (unique) capacity-achieving output distributions are denoted by $P_{Y_k}^*$. Finally, we also define the dispersions $V_k \triangleq V(P^*, W_k)$.

We are now ready to formally define a VLSF code for the broadcast channel with common message.

Definition 1: An (l, M, ϵ) -VLSF code for the broadcast channel with common message consists of:

- 1) A RV $U \in \mathcal{U}$, with $|\mathcal{U}| \leq 3$, which is known by the encoder and by both decoders.
- 2) A sequence of encoders $f_n : \mathcal{U} \times \mathcal{M} \rightarrow \mathcal{X}$, each one mapping the message $J \in \mathcal{M} = \{1, \dots, M\}$, drawn uniformly at random, to the channel input according to $X_n = f_n(U, J)$.
- 3) Two nonnegative integer-valued RVs τ_1 and τ_2 that are stopping times with respect to the filtrations $\mathcal{F}(U, Y_1^n)$ and $\mathcal{F}(U, Y_2^n)$, respectively, and which satisfy

$$\mathbb{E}[\max\{\tau_1, \tau_2\}] \leq l. \quad (6)$$

- 4) A sequence of decoders $g_{k,n} : \mathcal{U} \times \mathcal{Y}_k^n \rightarrow \mathcal{M}$ satisfying

$$\Pr[J \neq g_{k, \tau_k}(U, Y_k^{\tau_k})] \leq \epsilon, \quad k \in \{1, 2\}. \quad (7)$$

Remark 1: The RV U serves as common randomness, and enables the use of randomized codes [6]. To establish the cardinality bound on U , we proceed as in [3, Th. 19] to show that $|\mathcal{U}| \leq 4$ is sufficient. This bound can be further improved to $|\mathcal{U}| \leq 3$ by using the Fenchel-Eggleston theorem [7, p. 35].

Remark 2: VLSF codes require a feedback link from the decoders to the encoder. This feedback consists of a 1-bit stop signal per decoder, which is sent by decoder k at time τ_k . The encoder continuously transmits until both decoders have fed back a stop signal. Hence, the blocklength is $\max\{\tau_1, \tau_2\}$.

Our aim is to characterize the largest number of codewords $M_{\text{sf}}^*(l, \epsilon)$, whose average length is l , that can be transmitted with reliability $1 - \epsilon$ using a VLSF code.

III. MAIN RESULTS

A. Achievability bound

We first present an achievability bound. Its proof (omitted) follows closely the proof of [3, Th. 3].

Theorem 1: Fix $P \in \mathcal{P}(\mathcal{X})$. Let $\gamma_1, \gamma_2 \geq 0$ and $0 \leq q \leq 1$ be arbitrary scalars. Let the stopping times τ_k and $\bar{\tau}_k$, $k \in \{1, 2\}$, be defined as

$$\tau_k \triangleq \inf\{n \geq 0 : i_{P, W_k}(X^n; Y_k^n) \geq \gamma_k\} \quad (8)$$

$$\bar{\tau}_k \triangleq \inf\{n \geq 0 : i_{P, W_k}(\bar{X}^n; Y_k^n) \geq \gamma_k\} \quad (9)$$

where $(X^n, \bar{X}^n, Y_1^n, Y_2^n)$ are jointly distributed according to

$$\begin{aligned} & P_{X^n, \bar{X}^n, Y_1^n, Y_2^n}(x^n, \bar{x}^n, y_1^n, y_2^n) \\ &= P_{Y_1^n, Y_2^n|X^n}(y_1^n, y_2^n|x^n) \prod_{i=1}^n P(x_i)P(\bar{x}_i). \end{aligned} \quad (10)$$

For every M , there exists an (l, M, ϵ) -VLSF code such that

$$l \leq (1 - q)\mathbb{E}[\max\{\tau_1, \tau_2\}] \quad (11)$$

and

$$\epsilon \leq q + (1 - q)(M - 1)\Pr[\tau_k \geq \bar{\tau}_k]. \quad (12)$$

Remark 3: Following the same steps as in [3, Eq. (111)–(118)], ϵ in (12) can be further upper-bounded as

$$\epsilon \leq q + (1 - q)(M - 1)\exp\{-\gamma_k\}. \quad (13)$$

This bound is easier to evaluate and to analyze asymptotically.

B. Converse bound

Let $P_{\mathbf{x}^n} \in \mathcal{P}(\mathcal{X})$ be the type [8, Def. 2.1] of the sequence $\mathbf{x}^n \in \mathcal{X}^n$. We are now ready to state our converse bound.

Theorem 2: For every M , $t \in \mathbb{Z}_+$ and $\delta > 0$, let

$$\lambda_t \triangleq \log M - \log \log M - \delta - (|\mathcal{X}| - 1) \log(t + 1) \quad (14)$$

and let

$$\begin{aligned} L_t \triangleq & \prod_{k=1}^2 \max_{\mathbf{x}^t \in \mathcal{X}^t} \{\Pr[i_{P_{\mathbf{x}^t}, W_k}(\mathbf{x}^t; Y_k^t) > \lambda_t]\} \\ & + \epsilon_M \left(1 + \min_k \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[i_{P_{\mathbf{x}^t}, W_k}(\mathbf{x}^t; Y_k^t) > \lambda_t] \right) \end{aligned} \quad (15)$$

where $\varepsilon_M = \epsilon + (\log M)^{-1}$. Then, for every (l, M, ϵ) -VLSF code, we have

$$l \geq \sum_{t=0}^{\infty} (1 - L_t)^+ . \quad (16)$$

Proof: See Section IV. ■

C. Asymptotic expansion

Analyzing (13) and (16) in the limit $l \rightarrow \infty$, we obtain the following asymptotic characterization of $M_{\text{sf}}^*(l, \epsilon)$.

Theorem 3: Let $Z_k \sim \mathcal{N}(0, 1)$, $V = \sqrt{V_1 V_2}$, $\varrho_k = (V_k/V_k)^{1/4}$, and let $y = \tilde{Q}^{-1}(x)$ be the solution of

$$\prod_{k=1}^2 Q(-\varrho_k y) + x \left(1 + \min_k Q(-\varrho_k y) \right) = 1. \quad (17)$$

For every discrete memoryless broadcast channel with $C_1 = C_2$ and every $\epsilon \in (0, 1)$, we have

$$\begin{aligned} \frac{Cl}{1-\epsilon} - \Xi_a \sqrt{l} - \mathcal{O}(l^{1/4+\delta}) &\leq \log M_{\text{sf}}^*(l, \epsilon) \\ &\leq \frac{Cl}{1-\epsilon} - \Xi_c \sqrt{l} + \mathcal{O}(\log l) \end{aligned} \quad (18)$$

where $\delta > 0$ is an arbitrarily small constant,

$$\Xi_a \triangleq \sqrt{\frac{V_1 + V_2}{2\pi(1-\epsilon)}} \quad (19)$$

and

$$\begin{aligned} \Xi_c \triangleq & \sqrt{\frac{V}{(1-\epsilon)^3}} \left(\mathbb{E} \left[\min \left\{ \tilde{Q}^{-1}(\epsilon), \max_k \varrho_k Z_k \right\} \right] \right. \\ & \left. - \epsilon \left(2\tilde{Q}^{-1}(\epsilon) - \min_k \mathbb{E} \left[\min \left\{ \tilde{Q}^{-1}(\epsilon), \varrho_k Z_k \right\} \right] \right) \right). \end{aligned} \quad (20)$$

Proof: The converse bound in (18) is proved in Section V and the achievability bound is proved in Section VI. ■

Remark 4: When $C_1 \neq C_2$, it can be shown that the square-root penalty on the LHS of (18) vanishes. In this case, the problem reduces to the point-to-point transmission to the weakest decoder, for which the zero-dispersion result in [3] applies.

Remark 5: For the case when $P_{Y_{1,i}, Y_{2,i} | X_i}$ does not satisfy (3), a bound similar to the LHS of (18) can be obtained by replacing Ξ_a in (19) with

$$\sqrt{\frac{V_1 + V_2 - 2\text{Cov}(\imath_{P^*, W_1}(X; Y_1), \imath_{P^*, W_2}(X; Y_2))}{2\pi(1-\epsilon)}}. \quad (21)$$

Remark 6: When $\varrho_1 = \varrho_2 = 1$ (and, hence, $V_1 = V_2$), one can simplify the RHS of (18) as follows:

$$\begin{aligned} \log M_{\text{sf}}^*(l, \epsilon) &\leq \frac{Cl}{1-\epsilon} - \sqrt{\frac{Vl}{(1-\epsilon)^3}} \\ &\times \left(\frac{1}{\sqrt{\pi}} \left(1 - Q\left(\sqrt{2}Q^{-1}(\epsilon)\right) \right) + (\epsilon - 2)\phi(Q^{-1}(\epsilon)) \right) \\ &- \mathcal{O}(\log l). \end{aligned} \quad (22)$$

The second-order term in (22) is strictly negative for all $\epsilon \leq 0.1968$. This implies that, when $C_1 = C_2$, $V_1 = V_2$, and $\epsilon \leq 0.1968$, the asymptotic expansion of $\log M_{\text{sf}}^*(l, \epsilon)$ contains a square-root penalty.

IV. PROOF OF THEOREM 2

Fix M and ϵ . To establish Theorem 2, we derive a lower bound on l that holds for all VLSF codes having M codewords and probability of error no larger than ϵ . Since,

$$l \geq \mathbb{E}[\max\{\tau_1, \tau_2\}] = \sum_{t=0}^{\infty} (1 - \Pr[\max\{\tau_1, \tau_2\} \leq t]) \quad (23)$$

we can lower-bound l by upper-bounding $\Pr[\max\{\tau_1, \tau_2\} \leq t]$ for every $t \in \mathbb{Z}_+$. The following property (proven in Appendix I-A) turns out to be useful.

Property 1: Fix $t \in \mathbb{Z}_+$ and $\alpha \in [0, 1]$, and suppose there exists an (l, M, ϵ) -VLSF code with $\Pr[\max\{\tau_1, \tau_2\} \leq t] \leq \alpha$. Then there exists an (l', M, ϵ) -VLSF code for some $l' \geq l$, for which $\Pr[\max\{\tau_1, \tau_2\} \leq t] \leq \alpha$ and $\tau_1, \tau_2 \in \{t, t+1, \dots\}$.

Fix an arbitrary (l, M, ϵ) -VLSF code, defined by the tuple $(f_n, g_{1,n}, g_{2,n}, \tau_1, \tau_2, U)$. By Property 1, it is sufficient to consider codes for which $\tau_1, \tau_2 \in \{t, t+1, \dots\}$. Let $\epsilon_k^{(u)}$, $u \in \mathcal{U}$, be constants in $[0, 1]$ such that $\sum_{u \in \mathcal{U}} P_U(u) \epsilon_k^{(u)} \leq \epsilon$ and $\Pr[J \neq g_{k, \tau_k}(U, Y_k^{\tau_k}) | U = u] \leq \epsilon_k^{(u)}$.

Since $\{\tau_k = n\} \in \mathcal{F}(U, Y_k^n)$, we can define a sequence of binary functions $\varphi_k \triangleq \{\varphi_{k,t}, \varphi_{k,t+1}, \dots\}$ such that $\varphi_{k,n}(u, y_k^n) \triangleq \mathbb{1}\{\tau_k = n\}$. Let $P_{\mathbf{X}}^{(u)}$ be the conditional probability measure on \mathcal{X}^∞ induced by the encoder given $U = u$. Define for $u \in \mathcal{U}$ the set $\tilde{\mathcal{Y}}_k^{(u)} \triangleq \{y^n \in \mathcal{Y}_k^n : \varphi_{k,n}(u, y^n) = 1\}$. Note that we must have $Y_k^{\tau_k} \in \tilde{\mathcal{Y}}_k^{(u)}$. Let the length of a sequence of channel outputs $\bar{y} \in \tilde{\mathcal{Y}}_k^{(u)}$ be denoted by $|\bar{y}|$. On $\tilde{\mathcal{Y}}_k^{(u)}$, define the conditional probability measure $\mathbb{P}_{\bar{Y}|\mathbf{X}}^{(k,u)}$, given $\mathbf{x} \in \mathcal{X}^\infty$ and $u \in \mathcal{U}$, as

$$\mathbb{P}_{\bar{Y}|\mathbf{X}}^{(k,u)}(\bar{y}|\mathbf{x}) \triangleq \prod_{i=1}^{|\bar{y}|} W(\bar{y}_i|\mathbf{x}_i) \quad (24)$$

and the probability measure $\mathbb{P}_{\bar{Y}, \mathbf{X}}^{(k,u)}(\bar{y}, \mathbf{x}) \triangleq \mathbb{P}_{\bar{Y}|\mathbf{X}}^{(k,u)}(\bar{y}|\mathbf{x}) P_{\mathbf{X}}^{(u)}(\mathbf{x})$ on $\tilde{\mathcal{Y}}^{(u)} \times \mathcal{X}^\infty$. We also need the following auxiliary probability measure $\mathbb{Q}_{\bar{Y}}^{(k,u)}$ on $\tilde{\mathcal{Y}}_k^{(u)}$

$$\begin{aligned} \mathbb{Q}_{\bar{Y}}^{(k,u)}(\bar{y}) &\triangleq \\ &\sum_{P_{\mathbf{x}^t} \in \mathcal{P}_t(\mathcal{X})} \left(\frac{1}{|\mathcal{P}_t(\mathcal{X})|} \prod_{i=1}^t P_{\mathbf{x}^t} W_k(\bar{y}_i) \prod_{i=t+1}^{|\bar{y}|} P_{Y_k}^*(\bar{y}_i) \right) \end{aligned} \quad (25)$$

and the probability measure $\mathbb{Q}_{\bar{Y}, \mathbf{X}}^{(k,u)}(\bar{y}, \mathbf{x}) = \mathbb{Q}_{\bar{Y}}^{(k,u)}(\bar{y}) P_{\mathbf{X}}^{(u)}(\mathbf{x})$ on $\tilde{\mathcal{Y}}^{(u)} \times \mathcal{X}^\infty$. Here, $\mathcal{P}_t(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$ denotes the set of types formed by length- t sequences.

Using the meta-converse theorem [9, Th. 27], the inequality [9, Eq. (102)], the fact that $\mathbb{Q}_{\bar{Y}, \mathbf{X}}^{(k,u)}$ is a convex combination of distributions [10, Lem. 3], and the upper bound $|\mathcal{P}_t(\mathcal{X})| \leq (t+1)^{|\mathcal{X}|-1}$ [11, Lem. 1.1], we conclude that (see Appendix I-B)

$$\mathbb{P}_{\bar{Y}, \mathbf{X}}^{(k,u)}[\tilde{l}_k^{(u)}(\mathbf{X}; \bar{Y}_k) \leq \lambda_t] \leq \varepsilon_{k,M}^{(u)} \quad (26)$$

where $\varepsilon_{k,M}^{(u)} \triangleq \varepsilon_k^{(u)} + (\log M)^{-1}$ and λ_t is defined in (14). Here,

$$\hat{\imath}_k^{(u)}(\mathbf{x}; \bar{y}) \triangleq \imath_k(\mathbf{x}^t; y^t) + \sum_{i=t+1}^{\lfloor \bar{y} \rfloor} \log \frac{W_k(y_i | \mathbf{x}_i)}{P_{Y_k}^*(y_i)} \quad (27)$$

where $\imath_k(\mathbf{x}^t; y^t) \triangleq \imath_{P_{\mathbf{x}^t}, W_k}(\mathbf{x}^t, y^t)$. Next, we minimize $\Pr[\tau_k \leq t | U = u]$ over all stopping times τ_k satisfying (26):

$$\begin{aligned} \Pr[\tau_k \leq t | U = u] &= \mathbb{P}_{\bar{Y}, \mathbf{X}}^{(k,u)}[|\bar{Y}| = t] \\ &= \mathbb{P}_{\bar{Y}, \mathbf{X}}^{(k,u)}[\hat{\imath}_k^{(u)}(\mathbf{X}; \bar{Y}_k) > \lambda_t, |\bar{Y}| = t] \\ &\quad + \mathbb{P}_{\bar{Y}, \mathbf{X}}^{(k,u)}[\hat{\imath}_k^{(u)}(\mathbf{X}; \bar{Y}_k) \leq \lambda_t, |\bar{Y}| = t] \end{aligned} \quad (28)$$

$$\leq \min \left\{ 1, \mathbb{P}_{\bar{Y}, \mathbf{X}}^{(k,u)}[\hat{\imath}_k^{(u)}(\mathbf{X}; \bar{Y}_k) > \lambda_t, |\bar{Y}| = t] + \varepsilon_{k,M}^{(u)} \right\} \quad (29)$$

$$\begin{aligned} &\leq \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda_t] \\ &\quad + \min \left\{ \varepsilon_{k,M}^{(u)}, 1 - \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda_t] \right\}. \end{aligned} \quad (30)$$

Here, (29) follows from (26). Since the stopping times τ_1 and τ_2 are conditional independent given $U = u$, (30) implies that

$$\Pr[\max\{\tau_1, \tau_2\} \leq t | U = u] = \prod_{k=1}^2 \mathbb{P}_{\bar{Y}, \mathbf{X}}^{(k,u)}[|\bar{Y}_k| = t] \quad (31)$$

$$\begin{aligned} &\leq \prod_{k=1}^2 \max_{\mathbf{x}^t \in \mathcal{X}^t} \{\Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda_t]\} \\ &\quad + \min_k \left\{ \varepsilon_{k,M}^{(u)} + \varepsilon_{k,M}^{(u)} \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda_t] \right\}. \end{aligned} \quad (32)$$

Note that (32) holds for all τ_k that satisfies (26). Averaging (32) over $u \in \mathcal{U}$ and using the inequality $\sum_{u \in \mathcal{U}} P_U(u) \varepsilon_{k,M}^{(u)} \leq \epsilon + (\log M)^{-1} = \varepsilon_M$, we obtain (15). The proof is concluded using (23).

V. ASYMPTOTIC ANALYSIS: CONVERSE BOUND

We analyze L_t in (15) in the limit $l \rightarrow \infty$. By (16),

$$l \geq \sum_{t=0}^{\infty} (1 - L_t)^+ \geq \sum_{t=0}^{\lfloor \beta \rfloor} (1 - L_t)^+ \geq \sum_{t=0}^{\lfloor \beta \rfloor} (1 - L_t) \quad (33)$$

where $\beta > 0$ will be specified shortly. Let $\lambda \triangleq \log M - \log \log M - \delta - (|\mathcal{X}| - 1) \log(\beta + 1)$. For all $t \leq \beta$,

$$\max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda_t] \leq \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda]. \quad (34)$$

The key step is to establish an asymptotic upper bound on $\max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda]$ for every $t \in \mathbb{Z}_+$ as $\lambda \rightarrow \infty$.

Let $\alpha \triangleq \frac{\lambda}{C} - \sqrt{\frac{V\lambda}{C^3}} \log \lambda$ and let β be the solution of

$$(\lambda - \beta C) / \sqrt{\beta V} = -\tilde{Q}^{-1}(\epsilon) \quad (35)$$

where C is given in (5), V is defined in Theorem 3, and $\tilde{Q}^{-1}(\epsilon)$ in (17). We divide the asymptotic analysis of $\max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda]$ into three cases: the ‘‘large deviations regime’’ $t \in [0, \alpha]$, where we use Hoeffding’s inequality, the ‘‘central regime’’ $t \in [\alpha, \beta]$, where Berry-Esseen central

limit theorem is applied, and the case $t \geq \beta$, where the trivial upper bound $\max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda] \leq 1$ suffices.

In the first case, invoking Hoeffding’s inequality [12, Th. 2] and using that $I(P_{\mathbf{x}^t}, W_k)$ is upper-bounded by C uniformly, we obtain (see Appendix II-A for details)

$$\sum_{t=0}^{\lfloor \alpha \rfloor} \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda] = o(1), \quad \lambda \rightarrow \infty \quad (36)$$

and

$$\sum_{t=0}^{\lfloor \alpha \rfloor} \prod_{k=1}^2 \max_{\mathbf{x}^t \in \mathcal{X}^t} \{\Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda]\} = o(1), \quad \lambda \rightarrow \infty. \quad (37)$$

In the central regime, we use the Berry-Esseen central limit theorem [13, Th. V.3] to show that

$$\Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda] \leq Q\left(\frac{\lambda - tI(P_{\mathbf{x}^t}, W_k)}{\sqrt{tV(P_{\mathbf{x}^t}, W_k)}}\right) + \frac{c}{\sqrt{t}}. \quad (38)$$

We next maximize (38) over $\mathbf{x}^t \in \mathcal{X}^t$ following the approach in [10, Prop. 8]. Specifically, we use continuity properties of $I(P, W_k)$ and $V(P, W_k)$ for probability distributions $P \in \mathcal{P}(\mathcal{X})$ close to P^* to show that (see Appendix II-B)

$$\begin{aligned} &\sum_{t=\lfloor \alpha \rfloor + 1}^{\lfloor \beta \rfloor} \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda] \\ &\leq \sqrt{\frac{V\lambda}{C^3}} \left(\tilde{Q}^{-1}(\epsilon) - \mathbb{E} \left[\min \left\{ \tilde{Q}^{-1}(\epsilon), \varrho_k Z_k \right\} \right] \right) + \mathcal{O}(\log \lambda) \end{aligned} \quad (39)$$

where ϱ_k are defined in Theorem 3 and $Z_k \sim \mathcal{N}(0, 1)$. Similarly, we obtain

$$\begin{aligned} &\sum_{t=\lfloor \alpha \rfloor + 1}^{\lfloor \beta \rfloor} \prod_{k=1}^2 \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda] \\ &\leq \sqrt{\frac{V\lambda}{C^3}} \left(\tilde{Q}^{-1}(\epsilon) - \mathbb{E} \left[\min \left\{ \tilde{Q}^{-1}(\epsilon), \max_k \varrho_k Z_k \right\} \right] \right) \\ &\quad + \mathcal{O}(\log \lambda). \end{aligned} \quad (40)$$

Using (33), (36), (37), (39), and (40), we obtain

$$\begin{aligned} l &\geq \sum_{t=0}^{\lfloor \beta \rfloor} (1 - L_t) \\ &\geq \frac{\lambda(1 - \varepsilon_M)}{C} + \sqrt{\frac{V\lambda}{C^3}} \left(\mathbb{E} \left[\min \left\{ \tilde{Q}^{-1}(\epsilon), \max_k \varrho_k Z_k \right\} \right] \right. \\ &\quad \left. - \varepsilon_M \left(2\tilde{Q}^{-1}(\epsilon) - \min_k \mathbb{E} \left[\min \left\{ \tilde{Q}^{-1}(\epsilon), \varrho_k Z_k \right\} \right] \right) \right) \\ &\quad - \mathcal{O}(\log \lambda) \end{aligned} \quad (41)$$

$$\quad (42)$$

as $\lambda \rightarrow \infty$. Finally, we have that

$$\begin{aligned} \lambda &= \log M - \log \log M - \delta - (|\mathcal{X}| - 1) \log(\beta + 1) \quad (43) \\ &\leq \frac{Cl}{1 - \varepsilon_M} \\ &\quad - \sqrt{\frac{Vl}{(1 - \varepsilon_M)^3}} \left(\mathbb{E} \left[\min \left\{ \tilde{Q}^{-1}(\epsilon), \max_k \varrho_k Z_k \right\} \right] \right. \\ &\quad \left. - \varepsilon_M \left(2\tilde{Q}^{-1}(\epsilon) - \min_k \mathbb{E} \left[\min \left\{ \tilde{Q}^{-1}(\epsilon), \varrho_k Z_k \right\} \right] \right) \right) \\ &\quad + \mathcal{O}(\log l) \end{aligned} \quad (44)$$

as $l \rightarrow \infty$. The final result in (18) is obtained through algebraic manipulations.

VI. ASYMPTOTIC ANALYSIS: ACHIEVABILITY BOUND

Set $P = P^*$, and fix $r \in \mathbb{N}$, $q = \frac{l'\epsilon - 1}{l' - 1}$, and $l' > 0$, a parameter that will be related to the average blocklength. Let the thresholds be chosen as follows:

$$\gamma \triangleq \gamma_k \triangleq C(l' - g(Cl')). \quad (45)$$

Here,

$$g(x) \triangleq \sqrt{\frac{V_1 + V_2}{2\pi C^2}} \sqrt{\frac{x}{C}} + b_1 x^{\frac{r+1}{4r+2}} \log x \quad (46)$$

where b_1 will be specified later. If we choose a code with a number of codewords \tilde{M} that satisfies

$$\log \tilde{M} \triangleq C(l' - g(Cl')) - \log l' \quad (47)$$

we have $(\tilde{M} - 1) \exp\{-\gamma\} \leq 1/l'$. Furthermore, by Remark 3, the average probability of error is upper-bounded by

$$\begin{aligned} q + (1 - q)(\tilde{M} - 1) \exp\{-\gamma_k\} \\ \leq \frac{l'\epsilon - 1}{l' - 1} + \frac{l'(1 - \epsilon)}{l' - 1} \frac{1}{l'} = \epsilon. \end{aligned} \quad (48)$$

Suppose it can be shown that

$$\mathbb{E}[\max\{\tau_1, \tau_2\}] \leq l' \quad (49)$$

for sufficiently large l' . Then the average blocklength is

$$(1 - q)\mathbb{E}[\max\{\tau_1, \tau_2\}] \leq \frac{l'(1 - \epsilon)}{l' - 1} l' \triangleq l. \quad (50)$$

Consequently, by Theorem 1, there exists an (l, M, ϵ) -VLSF code with

$$\log M \geq \log \tilde{M} \quad (51)$$

$$= C(l' - g(Cl')) - \log l' \quad (52)$$

$$= \frac{Cl}{1 - \epsilon} - \sqrt{\frac{V_1 + V_2}{2\pi(1 - \epsilon)}} \sqrt{l} - \mathcal{O}(l^{\frac{r+1}{4r+2}} \log l) \quad (53)$$

where the last step follows because

$$l = \frac{(l')^2(1 - \epsilon)}{l' - 1} = l'(1 - \epsilon) + o(1). \quad (54)$$

To establish (49), we proceed as follows. Let $W_n = \imath_{P, W_1}(X_n; Y_{1,n})$ and $Z_n = \imath_{P, W_2}(X_n; Y_{2,n})$. We can then

upper-bound $\mathbb{E}[\max\{\tau_1, \tau_2\}]$ using the following lemma, which is proved in Appendix III.

Lemma 1: Let $\{W_n\}$ and $\{Z_n\}$, $n \geq 1$, be i.i.d. discrete RVs with $(W_1, Z_1) \sim P_{W,Z}$, positive mean $\mu_W \triangleq \mathbb{E}[W_1]$ and $\mu_Z \triangleq \mathbb{E}[Z_1]$, respectively, and finite moments of order $r \geq 3$, i.e., $\mathbb{E}[|W_1|^r] < \infty$, and $\mathbb{E}[|Z_1|^r] < \infty$. Define the random walks $U_n \triangleq \sum_{i=1}^n W_i$ and $V_n \triangleq \sum_{i=1}^n Z_i$, and the stopping times $\tau_1 \triangleq \inf\{n \geq 0 : U_n \geq \gamma\}$ and $\tau_2 \triangleq \inf\{n \geq 0 : V_n \geq \gamma\}$ for every $\gamma \in \mathbb{R}$. Then

$$\begin{aligned} \mathbb{E}[\max\{\tau_1, \tau_2\}] &\leq \frac{\gamma}{\min\{\mu_W, \mu_Z\}} + \frac{\sigma}{\sqrt{2\pi}} \sqrt{\frac{\gamma}{\mu_W}} \mathbb{1}\{\mu_W = \mu_Z\} \\ &\quad + \mathcal{O}\left(\gamma^{\frac{r+1}{4r+2}} \log \gamma\right) \end{aligned} \quad (55)$$

as $\gamma \rightarrow \infty$, where $\sigma^2 \triangleq \text{Var}\left[\frac{W_1}{\mu_W} - \frac{Z_1}{\mu_Z}\right]$.

Lemma 1 implies that there exists a constant b_1 such that

$$\mathbb{E}[\max\{\tau_1(\gamma), \tau_2(\gamma)\}] \leq \frac{\gamma}{C} + g(\gamma) \quad (56)$$

for sufficiently large γ . The conditional average blocklength of the VLSF code can be bounded as follows

$$\mathbb{E}[\max\{\tau_1, \tau_2\}] = \mathbb{E}[\max\{\tau_1(\gamma), \tau_2(\gamma)\}] \quad (57)$$

$$\leq \frac{\gamma}{C} + g(\gamma) \quad (58)$$

$$= l' - g(Cl') + g(Cl' - Cg(Cl')) \leq l'. \quad (59)$$

Here, (58) holds by (56), and (59) follows by the definition of γ in (45) and the fact that $g(x)$ is nonnegative and nondecreasing.

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APPENDIX I

STEPS OMITTED IN THE PROOF OF THE CONVERSE BOUND

A. Proof of Property 1

Let $(f_n, g_{1,n}, g_{2,n}, \tau_1, \tau_2, U)$ be a tuple defining an (l, M, ϵ) -VLSF code with $\Pr[\max\{\tau_1, \tau_2\} \leq t] \leq \alpha$. Set

$$\tilde{\tau}_k = \begin{cases} t, & \tau_k \leq t \\ \tau_k, & \tau_k > t \end{cases} \quad (60)$$

and

$$\tilde{g}_{k,n}(u, y_k^n) = \begin{cases} g_{k,n}(u, y_k^{\tau_k}), & \tau_k \leq n \\ g_{k,n}(u, y_k^n), & \tau_k > n. \end{cases} \quad (61)$$

Note that $\tilde{\tau}_k$ is also a stopping time with respect to the filtration $\mathcal{F}(U, Y_k^n)$ for $k \in \{1, 2\}$. Since τ_k is a function of U and Y_k^n given $\tau_k \leq n$, the new decoder $\tilde{g}_{k,n}$ is well-defined. Moreover, the decoders $g_{k,n}$ and $\tilde{g}_{k,n}$ yield the same probability of error. Thus $(f_n, \tilde{g}_{1,n}, \tilde{g}_{2,n}, \tilde{\tau}_1, \tilde{\tau}_2, U)$ defines an (l', M, ϵ) -VLSF code, with $l' \geq l$.

B. Proof of (26)

For each decoder k , the average probability of error is no larger than $\epsilon_k^{(u)}$ under $\mathbb{P}_{\bar{Y}_k, \mathbf{X}}^{(k,u)}$ and it is no larger than $1 - 1/M$ under $\mathbb{Q}_{\bar{Y}_k, \mathbf{X}}^{(k,u)}$. Hence, using the meta-converse theorem [9, Th. 27] and the inequality [9, Eq. (102)], we conclude that

$$\begin{aligned} \log M &\leq \log \tilde{\gamma}_k^{(u)} \\ &\quad - \log \left(\mathbb{P}_{\bar{Y}_k, \mathbf{X}}^{(k,u)} \left[\iota_k^{(u)}(\mathbf{X}; \bar{Y}_k) \leq \log \tilde{\gamma}_k^{(u)} \right] - \epsilon_k^{(u)} \right) \end{aligned} \quad (62)$$

for all $\tilde{\gamma}_k^{(u)}$ such that $\mathbb{P}_{\bar{Y}_k, \mathbf{X}}^{(k,u)} \left[\iota_k^{(u)}(\mathbf{X}; \bar{Y}_k) \leq \log \tilde{\gamma}_k^{(u)} \right] > \epsilon_k^{(u)}$. Here,

$$\iota_k^{(u)}(\mathbf{x}; \bar{y}_k) \triangleq \log \frac{\mathbb{P}_{\bar{Y}_k, \mathbf{X}}^{(k,u)}(\bar{y}_k, \mathbf{x})}{\mathbb{Q}_{\bar{Y}_k, \mathbf{X}}^{(k,u)}(\bar{y}_k, \mathbf{x})} = \log \frac{\mathbb{P}_{\bar{Y}_k | \mathbf{X}}^{(k,u)}(\bar{y}_k | \mathbf{x})}{\mathbb{Q}_{\bar{Y}_k}^{(k,u)}(\bar{y}_k)} \quad (63)$$

for all $\mathbf{x} \in \mathcal{X}^\infty$ and all $\bar{y}_k \in \mathcal{Y}_k^{(u)}$. Let now $\epsilon_{k,M}^{(u)} = \epsilon_k^{(u)} + (\log M)^{-1}$ and set $\tilde{\gamma}_k^{(u)} = \gamma_k^{(u)}$ where

$$\gamma_k^{(u)} \triangleq \sup \left\{ \nu \in \mathbb{R} : \mathbb{P}_{\bar{Y}_k, \mathbf{X}}^{(k,u)} \left[\iota_k^{(u)}(\mathbf{X}; \bar{Y}_k) \leq \log \nu \right] \leq \epsilon_{k,M}^{(u)} \right\}. \quad (64)$$

Note that there exists an arbitrary small positive constant δ , which is independent of $\log M$, such that

$$\begin{aligned} \mathbb{P}_{\bar{Y}_k, \mathbf{X}}^{(k,u)} \left[\iota_k^{(u)}(\mathbf{X}; \bar{Y}_k) \leq \log \gamma_k^{(u)} - \delta \right] \\ \leq \epsilon_{k,M}^{(u)} \leq \mathbb{P}_{\bar{Y}_k, \mathbf{X}}^{(k,u)} \left[\iota_k^{(u)}(\mathbf{X}; \bar{Y}_k) \leq \log \gamma_k^{(u)} \right]. \end{aligned} \quad (65)$$

Using (64) in (62), we obtain

$$\begin{aligned} \log M &\leq \log \gamma_k^{(u)} \\ &\quad - \log \left(\mathbb{P}_{\bar{Y}_k, \mathbf{X}}^{(k,u)} \left[\iota_k^{(u)}(\mathbf{X}; \bar{Y}_k) \leq \log \gamma_k^{(u)} \right] - \epsilon_k^{(u)} \right) \end{aligned} \quad (66)$$

$$\leq \log \gamma_k^{(u)} + \log \log M. \quad (67)$$

Finally, by (65) and (67), we have

$$\begin{aligned} \mathbb{P}_{\bar{Y}_k, \mathbf{X}}^{(k,u)} \left[\iota_k^{(u)}(\mathbf{X}; \bar{Y}_k) \leq \log M - \log \log M - \delta \right] \\ \leq \mathbb{P}_{\bar{Y}_k, \mathbf{X}}^{(k,u)} \left[\iota_k^{(u)}(\mathbf{X}; \bar{Y}_k) \leq \log \gamma_k^{(u)} - \delta \right] \end{aligned} \quad (68)$$

$$\leq \epsilon_{k,M}^{(u)}. \quad (69)$$

Using [10, Lem. 3] and the fact that $\mathbb{Q}_{\bar{Y}_k}^{(k,u)}$ is a convex combination of distributions, we obtain the following relation between $\iota_k^{(u)}(\mathbf{x}; \bar{y})$ and $\tilde{\iota}_k^{(u)}(\mathbf{x}; \bar{y})$

$$\iota_k^{(u)}(\mathbf{x}; \bar{y}) \leq \tilde{\iota}_k^{(u)}(\mathbf{x}; \bar{y}) - \log \frac{1}{|P_t(\mathcal{X})|}. \quad (70)$$

The inequality in (69) can then be rewritten using (70), as follows:

$$\begin{aligned} \epsilon_{k,M}^{(u)} &\geq \mathbb{P}_{\bar{Y}_k, \mathbf{X}}^{(k,u)} \left[\tilde{\iota}_k^{(u)}(\mathbf{X}; \bar{Y}_k) \leq \log M - \log \log M - \delta \right. \\ &\quad \left. - \log |P_t(\mathcal{X})| \right] \end{aligned} \quad (71)$$

$$\geq \mathbb{P}_{\bar{Y}_k, \mathbf{X}}^{(k,u)} \left[\tilde{\iota}_k^{(u)}(\mathbf{X}; \bar{Y}_k) \leq \lambda_t \right]. \quad (72)$$

Here, (72) follows by the definition of λ_t in (14), and because the number of types $|P_t(\mathcal{X})|$ is upper bounded by $(t+1)^{|\mathcal{X}-1|}$ [11, Lem. 1.1].

APPENDIX II

STEPS OMITTED IN THE ASYMPTOTIC ANALYSIS OF THE CONVERSE BOUND

We will need the following property, whose proof follows from standard algebraic manipulations.

Property 2: Fix arbitrary $x \in \mathbb{R}$, $a > 0$, $b > 0$, and $\lambda > 0$. Suppose that $\xi > 0$ is the unique solution to the equation

$$\frac{\lambda - \xi a}{\sqrt{b\xi}} = x. \quad (73)$$

Then

$$0 \leq \xi - \left(\frac{\lambda}{a} - x \sqrt{\frac{b\lambda}{a^3}} \right) \leq \frac{b}{a^2} x^2. \quad (74)$$

For notational convenience, we will denote the mean, variance and third absolute moment of $\iota_k(\mathbf{x}^t; Y_k^t)$ by

$$I_k(P_{\mathbf{x}^t}) \triangleq I(P_{\mathbf{x}^t}, W_k) \quad (75)$$

$$V_k(P_{\mathbf{x}^t}) \triangleq V(P_{\mathbf{x}^t}, W_k) \quad (76)$$

$$T_k(P_{\mathbf{x}^t}) \triangleq T(P_{\mathbf{x}^t}, W_k). \quad (77)$$

According to (74) and since β satisfies (35), we have

$$0 \leq \beta - \left(\frac{\lambda}{C} + \tilde{Q}^{-1}(\epsilon) \sqrt{\frac{V\lambda}{C^3}} \right) \leq \mathfrak{c}. \quad (78)$$

A. Proof of (36) and (37)

For the case $t < [0, \alpha]$, we use the following large-deviation bound

$$\begin{aligned} & \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[l_k(\mathbf{x}^t; Y_k^t) > \lambda] \\ & \leq \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr\left[\frac{l_k(\mathbf{x}^t; Y_k^t)}{t} - I_k(P_{\mathbf{x}^t}) \geq \frac{\lambda}{t} - I_k(P_{\mathbf{x}^t})\right] \end{aligned} \quad (79)$$

$$\leq \max_{\mathbf{x}^t \in \mathcal{X}^t} \exp\left(-\mathbb{C} \left(\frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{t}}\right)^2\right) \quad (80)$$

$$\leq \exp(-\mathbb{C} \log^2 \lambda) \quad (81)$$

$$\leq \left(\frac{1}{\lambda}\right)^{\mathbb{C} \log \lambda} \quad (82)$$

where (80) follows from Hoeffding's inequality [12, Th. 2] and (81) follows because $t < \alpha$ and because $I_k(P_{\mathbf{x}^t})$ is uniformly upper bounded by C . It follows from (82) that

$$\begin{aligned} & \sum_{t=0}^{[\alpha]} \max_{\mathbf{x}^t \in \mathcal{X}^\infty} \Pr[l_k(\mathbf{x}^t; Y_k^t) > \lambda] \\ & \leq (\alpha + 1) \left(\frac{1}{\lambda}\right)^{\mathbb{C} \log \lambda} \end{aligned} \quad (83)$$

$$\leq \mathbb{C} \left(\frac{1}{\lambda}\right)^{\mathbb{C} \log \lambda - 1} = o(1). \quad (84)$$

Using similar argument, one establishes (37).

B. Proof of (39) and (40)

For the case when $t \in [\alpha, \beta]$, we need tighter bounds on $I_k(P_{\mathbf{x}^t})$ and $V_k(P_{\mathbf{x}^t})$. Let Π_μ be the set of probability distributions that are at distance no larger than μ from P^* :

$$\Pi_\mu \triangleq \{P \in \mathcal{P}(\mathcal{X}) : \|P - P^*\|_2 \leq \mu\}. \quad (85)$$

Here, $\|P - P^*\|_2^2 \triangleq \sum_{x \in \mathcal{X}} (P(x) - P^*(x))^2$. Bounds on $I_k(P_{\mathbf{x}^t})$ and $V_k(P_{\mathbf{x}^t})$ are then supplied by [10, Lem. 7], which yields positive constants ς , μ and ρ for which

$$I_k(P_{\mathbf{x}^t}) \leq C - \varsigma \|P_{\mathbf{x}^t} - P^*\|_2^2 \quad (86)$$

$$V_k(P_{\mathbf{x}^t}) \geq \frac{V_k}{2} \quad (87)$$

and

$$|\sqrt{V_k(P_{\mathbf{x}^t})} - \sqrt{V_k}| \leq \rho \|P_{\mathbf{x}^t} - P^*\|_2 \quad (88)$$

for all $P_{\mathbf{x}^t} \in \Pi_\mu$.

Let $P_{\mathbf{x}^t} \in \Pi_\mu$. The Berry-Esseen central limit theorem yields the following estimate

$$\begin{aligned} & \Pr[l_k(\mathbf{x}^t; Y_k^t) > \lambda] \\ & \leq Q\left(\frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{tV_k(P_{\mathbf{x}^t})}}\right) + \frac{6tT_k(P_{\mathbf{x}^t})}{(tV_k(P_{\mathbf{x}^t}))^{3/2}} \end{aligned} \quad (89)$$

$$\leq Q\left(\frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{tV_k(P_{\mathbf{x}^t})}}\right) + \frac{\mathbb{C}}{\sqrt{t}} \quad (90)$$

where the last inequality follows from (87) and because $T_k(P_{\mathbf{x}^t}) < \mathbb{C}$ uniformly in Π_μ .

This also implies that for all $P_{\mathbf{x}^t} \in \Pi_\mu$,

$$\begin{aligned} & \max_{\mathbf{x}^t \in \mathcal{X}^t} \{\Pr[l_1(\mathbf{x}^t; Y_1^t) > \lambda]\} \max_{\mathbf{x}^t \in \mathcal{X}^t} \{\Pr[l_2(\mathbf{x}^t; Y_2^t) > \lambda]\} \\ & \leq \prod_{k=1}^2 \max_{\mathbf{x}^t \in \mathcal{X}^t} Q\left(\frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{tV_k(P_{\mathbf{x}^t})}}\right) + \frac{\mathbb{C}}{\sqrt{t}}. \end{aligned} \quad (91)$$

For the case when $P_{\mathbf{x}^t} \notin \Pi_\mu$, we use Chebyshev's inequality to obtain the estimate

$$\Pr[l_k(\mathbf{x}^t; Y_k^t) > \lambda] \leq \frac{tV_k(P_{\mathbf{x}^t})}{(\lambda - tI_k(P_{\mathbf{x}^t}))^2} \quad (92)$$

for all $\lambda > tI_k(P_{\mathbf{x}^t})$. Since $P_{\mathbf{x}^t} \notin \Pi_\mu$, there exists a constant C' such that $I_k(P_{\mathbf{x}^t}) \leq C' < C$. Hence, for sufficiently large λ , the condition $t \leq \beta$ implies that $\lambda > tI_k(P_{\mathbf{x}^t})$. Therefore, by (92), we have that

$$\begin{aligned} & \max_{P_{\mathbf{x}^t} \notin \Pi_\mu} \Pr[l_k(\mathbf{x}^t; Y_k^t) > \lambda] \\ & \leq \max_{P_{\mathbf{x}^t} \notin \Pi_\mu} \frac{tV_k(P_{\mathbf{x}^t})}{(\lambda - tI_k(P_{\mathbf{x}^t}))^2} \end{aligned} \quad (93)$$

$$\leq \frac{\mathbb{C}t}{(\lambda - tC')^2} \quad (94)$$

$$\leq \frac{\mathbb{C}\lambda}{(\lambda - \lambda C'/C - \mathbb{C}\sqrt{\lambda} - \mathbb{C})^2} \quad (95)$$

$$\leq \frac{\mathbb{C}}{\lambda} \quad (96)$$

where we have used that $t \leq \beta \leq 2t$ for sufficiently large λ and that $V_k(P_{\mathbf{x}^t})$ is uniformly upper-bounded [10, pp. 7048]. We see that $\max_{P_{\mathbf{x}^t} \notin \Pi_\mu} \Pr[l_k(\mathbf{x}^t; Y_k^t) > \lambda]$ can be driven arbitrarily close to zero by having λ sufficiently large. This implies that we only need to consider the input vectors \mathbf{x}^t for which $P_{\mathbf{x}^t} \in \Pi_\mu$, i.e.,

$$\begin{aligned} & \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[l_k(\mathbf{x}^t; Y_k^t) > \lambda] \\ & \leq \max_{P_{\mathbf{x}^t} \in \Pi_\mu} \Pr[l_k(\mathbf{x}^t; Y_k^t) > \lambda] + \frac{\mathbb{C}}{\lambda}. \end{aligned} \quad (97)$$

Using (90) and (97), we obtain

$$\begin{aligned} & \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[l_k(\mathbf{x}^t; Y_k^t) > \lambda] \\ & \leq \max_{P_{\mathbf{x}^t} \in \Pi_\mu} Q\left(\frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{tV_k(P_{\mathbf{x}^t})}}\right) + \frac{\mathbb{C}}{\sqrt{t}} + \frac{\mathbb{C}}{\lambda} \end{aligned} \quad (98)$$

$$\leq Q\left(\min_{P_{\mathbf{x}^t} \in \Pi_\mu} \frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{tV_k(P_{\mathbf{x}^t})}}\right) + \frac{\mathbb{C}}{\sqrt{t}} + \frac{\mathbb{C}}{t} \quad (99)$$

$$\begin{aligned} & = \int_{-\infty}^{\infty} \phi(x) \mathbb{1}\left\{\min_{P_{\mathbf{x}^t} \in \Pi_\mu} \frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{tV_k(P_{\mathbf{x}^t})}} \leq z\right\} dz + \frac{\mathbb{C}}{\sqrt{t}} \end{aligned} \quad (100)$$

for all sufficiently large λ . The indicator function in (100) can be upper bounded as

$$\begin{aligned} & \mathbb{1} \left\{ \min_{P_{\mathbf{x}^t} \in \Pi_\mu} \left\{ \frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{tV_k(P_{\mathbf{x}^t})}} - z \right\} \leq 0 \right\} \\ &= \mathbb{1} \left\{ \max_{P_{\mathbf{x}^t} \in \Pi_\mu} \left\{ tI_k(P_{\mathbf{x}^t}) + z\sqrt{tV_k(P_{\mathbf{x}^t})} - \lambda \right\} \geq 0 \right\} \end{aligned} \quad (101)$$

$$\leq \mathbb{1} \left\{ tC - t\varsigma\xi^2 + z\sqrt{tV_k} + |z|\sqrt{t\rho\xi} - \lambda \geq 0 \right\} \quad (102)$$

$$\leq \mathbb{1} \left\{ tC + z\sqrt{tV_k} + \frac{|z|\rho}{2\varsigma} - \lambda \geq 0 \right\} \quad (103)$$

$$\leq \mathbb{1} \left\{ \frac{\lambda - \frac{|z|\rho}{2\varsigma} - tC}{\sqrt{tV_k}} \leq z \right\} \quad (104)$$

$$\leq \mathbb{1} \left\{ \frac{\lambda}{C} - z\sqrt{\frac{\lambda V_k}{C^3}} - \frac{|z|\rho}{2C\varsigma} \leq t \right\} \quad (105)$$

where (101) follows since $\sqrt{tV_k(\mathbf{x}^t)} > 0$ for $P_{\mathbf{x}^t} \in \Pi_\mu$ by (87), (102) follows by (86) and (88) with $\xi \triangleq \|P_{\mathbf{x}^t} - P^*\|_2$, (103) follows because $-\varsigma\xi^2t + |z|\rho\xi\sqrt{t}$ is a quadratic expression in $\xi\sqrt{t}$ with maximum $\frac{|z|\rho}{2\varsigma}$ and (105) follows from (74). The steps (101)–(103) essentially follow from [10, Prop. 8]. Substituting (105) into (100) and summing from $\lfloor \alpha \rfloor + 1$ to $\lfloor \beta \rfloor$, we obtain

$$\begin{aligned} & \sum_{t=\lfloor \alpha \rfloor + 1}^{\lfloor \beta \rfloor} \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[i_k(\mathbf{x}^t; Y_k^t) > \lambda] \\ & \leq \sum_{t=0}^{\lfloor \beta \rfloor} \int_{-\infty}^{\infty} \phi(z) \mathbb{1} \left\{ \frac{\lambda}{C} - z\sqrt{\frac{V_k\lambda}{C^3}} - \frac{|z|\rho}{2C\varsigma} \leq t \right\} dz \\ & \quad + \mathcal{O}(\log \lambda) \end{aligned} \quad (106)$$

$$\begin{aligned} & \leq \int_0^\beta \int_{-\infty}^{\infty} \phi(z) \mathbb{1} \left\{ \frac{\lambda}{C} - z\sqrt{\frac{V_k\lambda}{C^3}} - \frac{|z|\rho}{2C\varsigma} \leq t \right\} dz dt \\ & \quad + \mathcal{O}(\log \lambda) \end{aligned} \quad (107)$$

$$\begin{aligned} & \leq \int_{-\infty}^{\infty} \phi(z) \int_0^\beta \mathbb{1} \left\{ \frac{\lambda}{C} - z\sqrt{\frac{V_k\lambda}{C^3}} - \frac{|z|\rho}{2C\varsigma} \leq t \right\} dt dz \\ & \quad + \mathcal{O}(\log \lambda) \end{aligned} \quad (108)$$

$$\leq \beta - \mathbb{E} \left[\min \left\{ \beta, \left(\frac{\lambda}{C} - Z_k \sqrt{\frac{V_k\lambda}{C^3}} \right) \right\} \right] + \mathcal{O}(\log \lambda) \quad (109)$$

$$\leq \sqrt{\frac{V\lambda}{C^3}} \left(\tilde{Q}^{-1}(\epsilon) - \mathbb{E} \left[\min \left\{ \tilde{Q}^{-1}(\epsilon), \varrho_k Z_k \right\} \right] \right) + \mathcal{O}(\log \lambda) \quad (110)$$

where ϱ_k are defined in Theorem 3 and $Z_k \sim \mathcal{N}(0, 1)$. Here, (107) follows because the indicator function is nondecreasing in t , in (108) the order of the integrals is interchangeable by Tonelli's theorem, and in (109) we have used (74).

By following the same approach, we obtain (40).

APPENDIX III PROOF OF LEMMA 1

Fix $\gamma \in \mathbb{R}$. We define the following two random walks, which are equivalent to U_n and V_n , but more convenient to analyze:

$$A_n \triangleq U_n/\mu_W + V_n/\mu_Z \quad (111)$$

$$B_n \triangleq U_n/\mu_W - V_n/\mu_Z. \quad (112)$$

We also define the additional stopping time

$$\tau_{12} \triangleq \inf \left\{ n \geq 0 : A_n \geq \gamma \frac{\mu_W + \mu_Z}{\mu_W \mu_Z} \right\}. \quad (113)$$

We shall next show that

$$\mathbb{E}[\max\{\tau_1, \tau_2\}] \leq \mathbb{E}[\tau_{12} + \tau'_1(\gamma - U_{\tau_{12}}) + \tau'_2(\gamma - V_{\tau_{12}})] \quad (114)$$

where $\tau'_1(\cdot)$ and $\tau'_2(\cdot)$ are defined as

$$\tau'_1(\tilde{\gamma}) = \inf \left\{ n \geq 0 : \sum_{i=1}^n \tilde{W}_i \geq \tilde{\gamma} \right\} \quad (115)$$

$$\tau'_2(\tilde{\gamma}) = \inf \left\{ n \geq 0 : \sum_{i=1}^n \tilde{Z}_i \geq \tilde{\gamma} \right\} \quad (116)$$

and where $\{\tilde{W}_k, \tilde{Z}_k\}$ are i.i.d. and $(\tilde{W}_1, \tilde{Z}_1) \sim P_{W,Z}$ but independent of W_j, Z_j for all $j \in \mathbb{N}$. Note that τ'_1 and τ'_2 are independent of $U_{\tau_{12}}$ and $V_{\tau_{12}}$.

To prove (114), we use the following argument. At time τ_{12} , we have that $U_{\tau_{12}}/\mu_W + V_{\tau_{12}}/\mu_Z \geq \gamma \frac{\mu_W + \mu_Z}{\mu_W \mu_Z}$. This implies that either $\tau_1 \leq \tau_{12}$ or $\tau_2 \leq \tau_{12}$ (or both) are satisfied. Consider the case $\tau_1 \leq \tau_{12}$ and $\tau_2 > \tau_{12}$. To bound $\mathbb{E}[\max\{\tau_1, \tau_2\}]$, we need to characterize the remaining time until the random walk V_n hits the threshold γ . This time is given by $\min\{n \geq 0 : V_{\tau_{12}+n} \geq \gamma\}$, which has the same distribution as (116) computed at $\gamma - V_{\tau_{12}}$. Note also that $\tau'_k(\tilde{\gamma}) = 0$ for every $\tilde{\gamma} \leq 0$ since we use the convention $\sum_{i=1}^0(\cdot) = 0$. The inequality in (114) follows because there exist events for which $\max\{\tau_1, \tau_2\} < \tau_{12}$. The case $\tau_2 \leq \tau_{12}$ and $\tau_1 > \tau_{12}$ can be analyzed similarly.

By [4, Th. 3.9.4] (or by Wald's equality when W_1 and Z_1 have bounded support [3, Eq. (106)–(107)]), we have

$$\frac{\tilde{\gamma}}{\mu_W} \leq \mathbb{E}[\tau'_1(\tilde{\gamma})] \leq \frac{\tilde{\gamma}}{\mu_W} + \mathfrak{c} \quad (117)$$

$$\frac{\tilde{\gamma}}{\mu_W} \leq \mathbb{E}[\tau'_2(\tilde{\gamma})] \leq \frac{\tilde{\gamma}}{\mu_Z} + \mathfrak{c} \quad (118)$$

$$\gamma \frac{\mu_W + \mu_Z}{2\mu_W \mu_Z} \leq \mathbb{E}[\tau_{12}] \leq \gamma \frac{\mu_W + \mu_Z}{2\mu_W \mu_Z} + \mathfrak{c}. \quad (119)$$

Using (114), the linearity of expectation, (117)–(119), and the fact that

$$\begin{aligned} \mathbb{E}[\tau'_1(\gamma - U_{\tau_{12}})] &= \mathbb{E}[\mathbb{E}[\tau'_1(\gamma - U_{\tau_{12}}) | U_{\tau_{12}}]] \\ &\leq \frac{1}{\mu_W} \mathbb{E}[(\gamma - U_{\tau_{12}})^+] + \mathfrak{c} \end{aligned} \quad (120)$$

we conclude that

$$\begin{aligned} & \mathbb{E}[\max\{\tau_1, \tau_2\}] - \gamma \frac{\mu_W + \mu_Z}{2\mu_W\mu_Z} \\ & \leq \frac{1}{\mu_W} \mathbb{E}[(\gamma - U_{\tau_{12}})^+] + \frac{1}{\mu_Z} \mathbb{E}[(\gamma - V_{\tau_{12}})^+] + \mathfrak{c} \quad (121) \end{aligned}$$

$$\begin{aligned} & = \frac{1}{\mu_W} \mathbb{E}\left[\left(\gamma - \frac{1}{2}\mu_W(A_{\tau_{12}} + B_{\tau_{12}})\right)^+\right] \\ & \quad + \frac{1}{\mu_Z} \mathbb{E}\left[\left(\gamma - \frac{1}{2}\mu_Z(A_{\tau_{12}} - B_{\tau_{12}})\right)^+\right] + \mathfrak{c} \quad (122) \end{aligned}$$

$$\begin{aligned} & \leq \mathbb{E}\left[\left(\frac{\gamma}{\mu_W} - \frac{1}{2}\left(\gamma \frac{\mu_W + \mu_Z}{\mu_W\mu_Z} + B_{\tau_{12}}\right)\right)^+\right] \\ & \quad + \mathbb{E}\left[\left(\frac{\gamma}{\mu_Z} - \frac{1}{2}\left(\gamma \frac{\mu_W + \mu_Z}{\mu_W\mu_Z} - B_{\tau_{12}}\right)\right)^+\right] + \mathfrak{c} \quad (123) \end{aligned}$$

$$= \frac{1}{2} \mathbb{E}\left[\left|\gamma \frac{\mu_Z - \mu_W}{\mu_W\mu_Z} - B_{\tau_{12}}\right|\right] + \mathfrak{c} \quad (124)$$

where (123) follows from the definition of τ_{12} (see (113)) which implies that $A_{\tau_{12}} \geq \gamma \frac{\mu_W + \mu_Z}{\mu_W\mu_Z}$.

We next show that the RHS of (124) is upper-bounded by the RHS of (55) by the following two steps. First, we shall approximate $B_{\tau_{12}}$ by a Gaussian RV using a variation of the Berry-Esseen theorem that holds when the number of terms in the summation is a RV (see Lemma 2 below). Then, we shall establish (55) using standard properties of Gaussian RVs.

Lemma 2: ([5, Th. 1]) Let $\{\xi_n, n \geq 1\}$ be i.i.d. RVs with zero mean, positive variance σ^2 , and finite third absolute moment. Let $\{N_n, n \in \mathbb{N}\}$ be a sequence of positive integer-valued RVs and assume that

$$\Pr\left[\left|\frac{N_n}{n\nu} - 1\right| > \zeta_n\right] = \mathcal{O}(\sqrt{\zeta_n}) \quad (125)$$

for some constant ν and a sequence $\{\zeta_n\}$ that vanishes as $n \rightarrow \infty$ and that satisfies $\frac{1}{n} \leq \zeta_n$ for all n . Then

$$\sup_{\lambda \in \mathbb{R}} \left| \Pr\left[\sum_{i=1}^{N_n} \xi_i \leq \sigma\sqrt{n\nu}\lambda\right] - \Phi(\lambda) \right| = \mathcal{O}(\sqrt{\zeta_n}). \quad (126)$$

The RV $B_{\tau_{12}}$ and its variance satisfies [4, Th. 4.2.4 (ii')]

$$\text{Var}[B_{\tau_{12}}] = \sigma^2 \gamma \frac{\mu_W + \mu_Z}{2\mu_W\mu_Z} + \mathcal{O}(1) \quad (127)$$

as $\gamma \rightarrow \infty$. For some constant $\nu > 0$, let $\gamma_n \triangleq \frac{2\nu\mu_W\mu_Z n}{\mu_W + \mu_Z}$, $N_n \triangleq \tau_{12}(\gamma_n)$, $\xi_n \triangleq \frac{W_n}{\mu_W} - \frac{Z_n}{\mu_Z}$ and $\zeta_n \triangleq n^{-\frac{r}{2r+1}}$ for $n \in \mathbb{N}$. Note that by (119), we have

$$\mathbb{E}[N_n] = \mathbb{E}[\tau_{12}(\gamma_n)] \quad (128)$$

$$= \gamma_n \frac{\mu_W + \mu_Z}{2\mu_W\mu_Z} + \mathcal{O}(1) \quad (129)$$

$$= \nu n + \mathcal{O}(1), \quad n \rightarrow \infty. \quad (130)$$

We next show that condition (125) in Lemma 2 is satisfied. Indeed,

$$\begin{aligned} & \Pr\left[\left|\frac{N_n}{\nu n} - 1\right| \geq \zeta_n\right] \\ & = \Pr\left[\left|\frac{N_n - \nu n}{\sqrt{\nu n}}\right|^r \geq (\sqrt{\nu n}\zeta_n)^r\right] \quad (131) \end{aligned}$$

$$\leq \frac{\mathbb{E}\left[\left|\frac{N_n - \nu n}{\sqrt{\nu n}}\right|^r\right]}{(\sqrt{\nu n}\zeta_n)^r} \quad (132)$$

$$= \frac{\mathfrak{c}}{(\sqrt{\nu n}\zeta_n)^r} \quad (133)$$

$$= \frac{\mathfrak{c}}{n^{r/2} (n^{-\frac{r}{2r+1}})^r} = \frac{\mathfrak{c}}{n^{\frac{r}{4r+2}}} = \mathcal{O}(\sqrt{\zeta_n}) \quad (134)$$

as $n \rightarrow \infty$. Here, (132) follows from Markov's inequality and (133) follows from [4, Th. 3.8.4(i)].

Let $F(\lambda) \triangleq \Pr[B_{N_n} \leq \sigma\sqrt{\nu n}\lambda]$. We can now use Lemma 2, which for sufficiently large n implies that

$$\sup_{\lambda \in \mathbb{R}} |F(\lambda) - \Phi(\lambda)| \leq \mathfrak{c} n^{-\frac{r}{4r+2}}. \quad (135)$$

We next refine our estimate in (135) using Lemma 3 below.

Lemma 3: ([13, Th. 9]) Let $F(x)$ be the cumulative distribution function of a RV that has finite moment of order p . Suppose that $0 < \Delta \triangleq \sup_x |F(x) - \Phi(x)| \leq 1/\sqrt{e}$. Then there exists a constant C_p , that depends only on p , such that

$$|F(x) - \Phi(x)| \leq \frac{C_p \Delta (\log \frac{1}{\Delta})^{p/2} + \rho_p}{1 + |x|^p} \quad (136)$$

for all x . Here

$$\rho_p = \left| \int_{-\infty}^{\infty} |x|^p dF(x) - \int_{-\infty}^{\infty} |x|^p d\Phi(x) \right|. \quad (137)$$

Using Lemma 3 and (135), we have that

$$|F(\lambda) - \Phi(\lambda)| \leq \frac{\mathfrak{c} n^{-\frac{r}{4r+2}} \log n + \rho_2(n)}{1 + \lambda^2}. \quad (138)$$

for $\lambda \in \mathbb{R}$ and sufficiently large n . Here,

$$\rho_2(n) = \left| \frac{\text{Var}[B_{N_n}]}{\sigma^2 n \nu} - 1 \right| = \left| \frac{n + \mathcal{O}(1)}{n} - 1 \right| \leq \frac{\mathfrak{c}}{n}. \quad (139)$$

Fix an arbitrary $a \in \mathbb{R}$. Using (138), we obtain the following upper bound

$$\begin{aligned} & \mathbb{E}[|a - B_{N_n}|] \\ & = \sigma\sqrt{\nu n} \int_0^\infty 1 + F\left(\frac{a}{\sigma\sqrt{\nu n}} - x\right) - F\left(\frac{a}{\sigma\sqrt{\nu n}} + x\right) dx \\ & \leq \sigma\sqrt{\nu n} \int_0^\infty \left[\Phi\left(\frac{a}{\sigma\sqrt{\nu n}} - x\right) + \left(1 - \Phi\left(\frac{a}{\sigma\sqrt{\nu n}} + x\right)\right) \right. \\ & \quad \left. + \frac{\mathfrak{c} n^{-\frac{r}{4r+2}} \log n + \mathfrak{c}/n}{1 + (\frac{a}{\sigma\sqrt{\nu n}} - x)^2} + \frac{\mathfrak{c} n^{-\frac{r}{4r+2}} \log n + \mathfrak{c}/n}{1 + (\frac{a}{\sigma\sqrt{\nu n}} + x)^2} \right] dx \\ & = \sigma\sqrt{\nu n} \mathbb{E}\left[\left|\frac{a}{\sigma\sqrt{\nu n}} - Z\right|\right] \quad (140) \end{aligned}$$

$$+ \pi \sigma \sqrt{\nu} \left(\mathfrak{C} n^{\frac{1}{2} - \frac{r}{4r+2}} \log n + \mathfrak{C} / \sqrt{n} \right) \quad (142)$$

$$= \sqrt{\frac{2}{\pi}} \sigma \sqrt{\nu n} \psi \left(\frac{a}{\sigma \sqrt{\nu n}} \right) + |a| + \mathcal{O}(n^{\frac{r+1}{4r+2}} \log n) \quad (143)$$

as $n \rightarrow \infty$, where $Z \sim \mathcal{N}(0, 1)$ and

$$\psi(x) \triangleq \sqrt{\frac{\pi}{2}} (\mathbb{E}[|x - Z|] - |x|) \quad (144)$$

$$= \exp \left(-\frac{x^2}{2} \right) + x \sqrt{\frac{\pi}{2}} \left(\operatorname{erf} \left(\frac{x}{\sqrt{2}} \right) - \operatorname{sgn}(x) \right). \quad (145)$$

The positive function $\psi(x)$ is unimodal with maximum 1 attained at $x = 0$ and decays exponentially to 0 as $|x| \rightarrow \infty$.

Substituting $a = \gamma_n \frac{\mu_Z - \mu_W}{\mu_W \mu_Z} = \frac{2\nu n(\mu_Z - \mu_W)}{\mu_W + \mu_Z}$ into (143), we obtain

$$\begin{aligned} & \mathbb{E} \left[\left| \gamma_n \frac{\mu_Z - \mu_W}{\mu_W \mu_Z} - B_{N_n} \right| \right] \\ & \leq \sqrt{\frac{2}{\pi}} \sigma \sqrt{\nu n} \psi \left(\frac{2\sqrt{\nu n}(\mu_Z - \mu_W)}{\sigma(\mu_W + \mu_Z)} \right) \\ & \quad + \gamma_n \left| \frac{\mu_Z - \mu_W}{\mu_W \mu_Z} \right| + \mathcal{O}(n^{\frac{r+1}{4r+2}} \log n). \end{aligned} \quad (146)$$

Note that for the case $\mu_Z \neq \mu_W$, we have that $\sqrt{n} \psi \left(\frac{2\sqrt{\nu n}(\mu_Z - \mu_W)}{\sigma(\mu_W + \mu_Z)} \right) = o(1)$ as $n \rightarrow \infty$. Substituting (146) into (124), we obtain

$$\begin{aligned} & \mathbb{E}[\max\{\tau_1(\gamma_n), \tau_2(\gamma_n)\}] \\ & \leq \gamma_n \frac{\mu_W + \mu_Z}{2\mu_W \mu_Z} \\ & \quad + \frac{1}{2} \mathbb{E} \left[\left| \gamma_n \frac{\mu_Z - \mu_W}{\mu_W \mu_Z} - B_{\tau_{12}(\gamma_n)} \right| \right] + \mathcal{O}(1) \end{aligned} \quad (147)$$

$$\begin{aligned} & = \gamma_n \frac{\mu_W + \mu_Z}{2\mu_W \mu_Z} + \gamma_n \left| \frac{\mu_Z - \mu_W}{2\mu_W \mu_Z} \right| \\ & \quad + \frac{\sigma}{\sqrt{2\pi}} \sqrt{\nu n} \mathbb{1}\{\mu_W = \mu_Z\} + \mathcal{O}(n^{\frac{r+1}{4r+2}} \log n) \end{aligned} \quad (148)$$

$$\begin{aligned} & = \frac{\gamma_n}{\min\{\mu_W, \mu_Z\}} + \frac{\sigma}{\sqrt{2\pi}} \sqrt{\nu n} \mathbb{1}\{\mu_W = \mu_Z\} \\ & \quad + \mathcal{O}(n^{\frac{r+1}{4r+2}} \log n), \quad n \rightarrow \infty \end{aligned} \quad (149)$$

where (149) follows from the identity $a + b + |a - b| = 2 \max\{a, b\}$.

To complete the proof, let $n_1 \triangleq \lceil \frac{\gamma}{\min(\mu_W, \mu_Z)} \rceil$, $\Psi(x) \triangleq x + \frac{\sigma}{\sqrt{2\pi}} \sqrt{\nu x} \mathbb{1}\{\mu_W = \mu_Z\} + b_1 x^{\frac{r+1}{4r+2}} \log x$, and set $\nu \triangleq \frac{\mu_W + \mu_Z}{2 \max\{\mu_W, \mu_Z\}}$, i.e.

$$\gamma_n = \min\{\mu_W, \mu_Z\} n. \quad (150)$$

Note that $\Psi(x)$ is nondecreasing, concave and differentiable in $x \in [1, \infty]$. Then there exists a constant $b_1 > 0$ such that

$$\begin{aligned} & \mathbb{E}[\max\{\tau_1(\gamma), \tau_2(\gamma)\}] \\ & \leq \mathbb{E}[\max\{\tau_1(\gamma_{n_1}), \tau_2(\gamma_{n_1})\}] \end{aligned} \quad (151)$$

$$\leq n_1 + \frac{\sigma}{\sqrt{2\pi}} \sqrt{\nu n_1} \mathbb{1}\{\mu_W = \mu_Z\} + b_1 n_1^{\frac{r+1}{4r+2}} \log n_1 \quad (152)$$

$$= \Psi \left(\left\lceil \frac{\gamma}{\min(\mu_W, \mu_Z)} \right\rceil \right) \quad (153)$$

$$\leq \Psi \left(\frac{\gamma}{\min\{\mu_W, \mu_Z\}} + 1 \right) \quad (154)$$

$$\leq \Psi \left(\frac{\gamma}{\min(\mu_W, \mu_Z)} \right) + \mathfrak{C} \quad (155)$$

$$\begin{aligned} & = \frac{\gamma}{\min\{\mu_W, \mu_Z\}} + \frac{\sigma}{2\sqrt{\pi}} \sqrt{\frac{\gamma(\mu_W + \mu_Z)}{\mu_W \mu_Z}} \mathbb{1}\{\mu_W = \mu_Z\} \\ & \quad + \mathcal{O}(\gamma^{\frac{r+1}{4r+2}} \log \gamma) \end{aligned} \quad (156)$$

$$\begin{aligned} & = \frac{\gamma}{\min\{\mu_W, \mu_Z\}} + \frac{\sigma}{\sqrt{2\pi}} \sqrt{\frac{\gamma}{\mu_W}} \mathbb{1}\{\mu_W = \mu_Z\} \\ & \quad + \mathcal{O}(\gamma^{\frac{r+1}{4r+2}} \log \gamma). \end{aligned} \quad (157)$$

Here, (151) follows because $\mathbb{E}[\max\{\tau_1(\gamma), \tau_2(\gamma)\}]$ is nondecreasing in γ .